

Discrepancy potential as a lens to designing and researching technology-based mathematics education tasks¹

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Abstract

It is unavoidable that a computer program has its constraints. In a mathematics pedagogical program such as dynamic geometry software (DGS), the constraints may lead to deviations of mathematical representations from its intended mathematical concepts. If it is used properly, the deviations can open up pedagogical opportunities and hence may be called as discrepancy potentials of the tool. In this paper, examples of developing mathematical discussions by using the discrepancy potentials in two DGS tasks are discussed. It aims at elaborating the idea of discrepancy potentials in the design principles for effective DGS-based tasks. This paper relates to the topic 'Mathematics Education using Information Communication Technology'.

1. Introduction

Dynamic geometry software (DGS) such as *GeoGebra* is a useful pedagogical tool because it can enable the users to manipulate mathematical objects. In the process of manipulation, experimental aspect and theoretical aspect of mathematics are interplayed [1]. The interplay can have potentials to enhance understanding of mathematical concepts and statements. Therefore, an important pedagogy research agenda is to investigate design principles for effective DGS-based tasks which can optimize the potential of experimental-theoretical interplay. One of the features of a tool-based task is that the mathematical representation produced by the tool (no matter it is software tool or concrete tool) can be different from the mathematical object itself. If we interpret a tool-based task as an experimentation of mathematical representation for theorizing mathematical objects, we may regard this discrepancy as a kind of experimental-theoretical interplay in tool-based tasks.

Inspired by a research study on designing and implementing mathematics tasks for teaching rotational symmetry by using a physical tool and a digital tool [2, 3, 4], the notion *discrepancy potential* is evolved. This paper aims at elaborating this notion and showcasing how this notion can be used to guide mathematical-rich discussions in classroom teaching.

In the following sections, the ideas of the notion discrepancy potential will be summarized. Then, how this notion can be used to guide mathematical discussions will be illustrated by two (hypothetical) teaching scenarios initiated from two *GeoGebra* files. Although DGS is used in these examples, it is believed that this notion is also useful for guiding the design of mathematics pedagogy tasks through other kinds of computer technologies. Last but not the least, research agenda related to this notion will be discussed.

2. Theorizing the notion of discrepancy potential

The notion discrepancy potential is inspired by a pedagogy research on using tool-based tasks to teach the concept of rotational symmetry. In that study, a lesson carried out with a tailor-made physical tool [2] was compared with one carried out with a digital tool [4]. It was found that the

¹ This is a revised and expanded version of the paper entitled *two examples on discrepancy potentials of dynamic geometry software* presented in the 21st Asian Technology Conference in Mathematics (ATCM2016).

concepts represented by these two tools deviated from the mathematical concepts intended to be taught. In one of the cases (the case of physical tool), the teacher used the discrepancy to open up a meaningful mathematical discussion which has extended the students' knowledge development. In another case (the case of digital tool), the teacher avoided the discrepancy which had potential to hinder students' understanding of the intended mathematical concept of that lesson. Based on this finding, the role of the deviation of the tool representations (no matter the tool is physical or digital) from the mathematical concepts intended to be taught is highlighted. The notion discrepancy potential is raised. It is defined as below.

The discrepancy potential of a tool is a pedagogical space generated by (i) feedback due to the nature of the tool or design of the task that possibly deviates from the intended mathematical concept or (ii) uncertainty created due to the nature of the tool or design of the task that requires the tool users to make decisions. [3] (p. 212)

As evidenced from the study mentioned above, discrepancy has potential to influence students' learning either positively or negatively in which the former is called as *discrepancy opportunity* and the latter as *discrepancy pitfall* [4] (p. 280). Yet, it is just a potential because whether it can actually influence students' learning depends on how the teacher handle this discrepancy which is influenced by how he or she perceives mathematics, the tool and the task assigned to the students. As what is pointed out in [4], "What counts as discrepancy is a relative notion which depends on the *perceived* nature of the chosen tool" (p. 276). In order to distinguish between the potential effect and the actual effect of discrepancy towards students' learning, two levels of discrepancy are proposed.

The first level is of *embedded discrepancy*. It is the tool's not being able to perform certain actions, a feature embedded in the tool per se. However, it has *potential* contribution to the affordances and constraints of the tool [...] which may be conducive to teaching and learning. The second level of discrepancy is *enacted discrepancy*. This type of discrepancy contributes *actually* to the affordances and constraints of the tool via *feedback* given to the user by the embedded discrepancy and hence it has actual influences on the emergence of mathematical knowledge. [4] (p. 276)

As evidenced in [2, 4], teacher's initiation of mathematics discussion based on discrepancy potential of a tool can lead to deep understanding and hence making the embedded discrepancy into an enacted one.

The notion of discrepancy potential is closely related to other theoretical perspectives of technology-based mathematics education tasks, including instrumental approach [5, 6], semiotic mediation [7], variation theory [8] and epistemic model of task design [9]. In order to avoid readers drown into the sea of theories, interested readers are referred to [4] for details. In this paper, we will elaborate the pedagogical potentiality of mathematics discussions based on tool discrepancies.

3. Using the notion of discrepancy potential to guide the design of mathematics pedagogy tasks

In the following, we will describe two cases of teaching tasks by using GeoGebra. It highlights the importance of incorporating mathematical discussions based on discrepancy potential embedded in DGS. The two tasks are initially designed by two pre-service mathematics teachers Jane and Peter (pseudonyms) respectively. They were end-of-term assignments of a mathematics pedagogy course which focuses on integrating computer technologies in mathematics classrooms. The pre-service

teachers were asked to design a self-contained teaching capsule of a mathematics topic with GeoGebra activities incorporated. In these two cases, there were some technical errors in the GeoGebra files. However, the present author (their course instructor) thinks that these technical errors can be used to enhance deep mathematical understanding. In the following subsections, the designs and the technical errors of these two files will be described. Then, the author will elaborate how he would use these technical errors to initiate mathematical discussions.

3.1 Case 1: Trigonometric functions

The topic of Jane's lesson design was trigonometric functions. She created a GeoGebra file [S1] to guide the students understand the definition of trigonometric ratios of angles in a rectangular plane and explore the signs of the trigonometric ratios and the reference angles. The GeoGebra file contains the standard unit circle representation of trigonometric functions. The input box and the slider enable the students to control the value of $\angle\theta$. As showed in Figure 1, $A=(0,0)$ is the center of the unit circle. $B=(1,0)$ is the intersection point of x -axis and the circle. B' is the rotated point of Point $B=(1,0)$ by $\angle\theta$. Point C is the perpendicular foot of B' on the x -axis. The auxiliary triangle $B'CA$ defines the trigonometric functions of $\angle\theta$: $\sin\theta = b = y(B')$ (that is, the y -coordinate of B'); $\cos\theta = a = x(C)$ (that is, the x -coordinate of C); and $\tan\theta = \frac{b}{a}$.

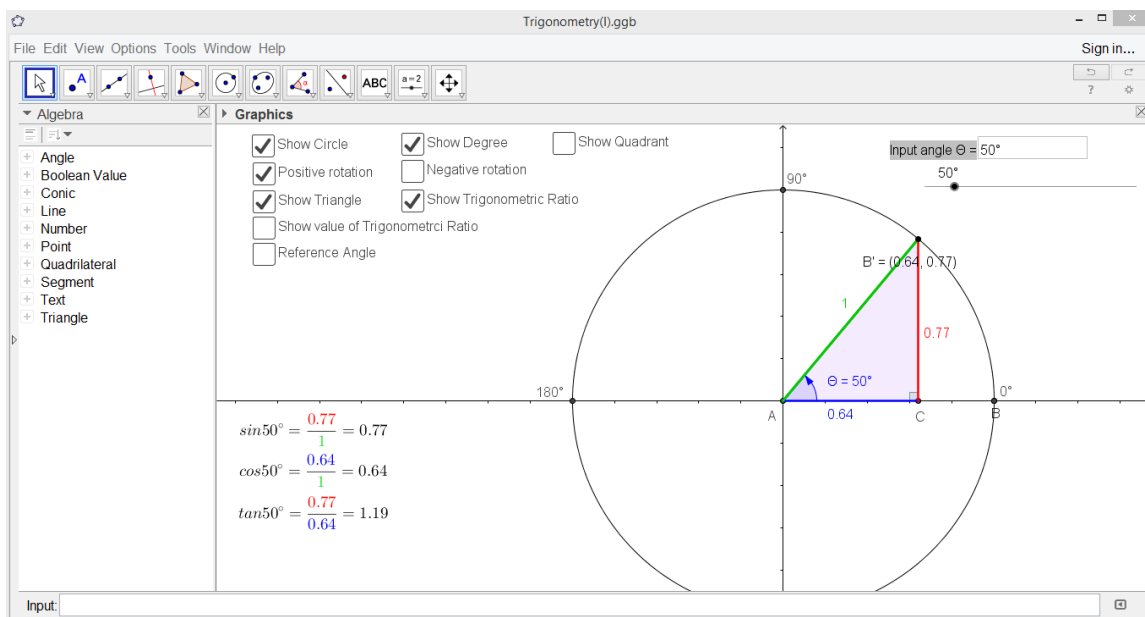


Figure 1: Visual representation of the definitions of trigonometric functions by coordinates

This dynamic figure provides a good visual representation of the definitions of trigonometric functions by coordinates. However, if we set $\angle\theta = 90^\circ$ (by inputting the value to the input box or dragging the slider to this value), we would find that $\sin\theta = 1$ and $\cos\theta = 0$ which are correctly displayed; but $\tan\theta$ displays a very large number (instead of the symbol ∞). This phenomenon is caused by the rounding-off error of the software. If we set rounding as 15 significant figures, we will see that $\cos\theta$ is a very small number (instead of 0) and hence $\tan\theta$ is a very large number (Figure 2). At this moment, we would be able to understand the source of the mistake, namely, point B' is rotated by an angle approximately equal to 90° (but not exact). This kind of error is frequently found in computer software. One should note that this rounding-off error would not happen if we set b as the length of line segment $B'C$ and a as the length of line segment AC . That

is, a and b are segments (instead of numbers) [S2]. In other words, this error was not due to the software design per se but how the user (the pre-service teacher) constructed the figure by using this software. However, we would argue that this error can be used to open up a learning opportunity. In the theory of discrepancy potential, this error is termed as an embedded discrepancy which has potential to provide an opportunity for students to enhance their conceptual understanding. However, it is only a potential because this discrepancy can actually contribute to conceptual understanding (i.e., the embedded discrepancy is transformed to an enacted discrepancy) only if the teacher can make use of it to initiate meaningful mathematics discussions.

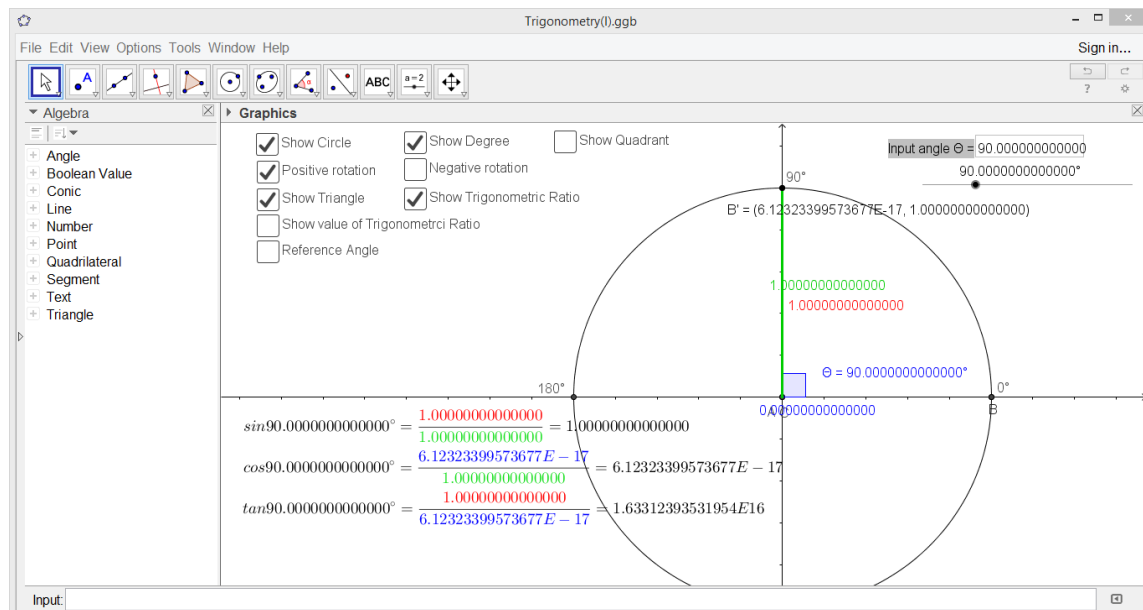


Figure 2: The representation of trigonometric functions at 90 degrees

In the following, we will propose how such kind of mathematics discussions can be arranged. First, with the rounding set back to the default value (2 decimal places) and θ set to 90° , the teacher can point out that the software shows $\tan 90^\circ = \frac{1}{0} = 16331239353195370$ and ask the students whether they think that it is correct. The teacher may remind the students that $\tan \theta = \frac{\sin \theta}{\cos \theta}$ and then prompt them to think about the result when 1 is divided by 0. The latter problem is known to be a difficult question for school students. The teacher may ask them to multiply 0 on both sides. They should be able to realize the absurdity. Yet, the teacher can further discuss with the students the product of a very large number multiplied by 0. It may lead to discussion on the intuitive meaning of $\infty \times 0$. After that, the teacher may discuss with the students why a large number is displayed for $\tan 90^\circ$. Then, the rounding may be set to a more precise value, say 15 significant figures. As such, $x(C)$ becomes a very small number (instead of zero) and $\tan 90^\circ$ becomes $\frac{1}{\text{small number}} = \text{large number}$. (See Figure 2 above.) The teacher may further discuss with the students the product of a small number and a large number in which an intuitive idea of the notion of indeterminate in pre-calculus would be provided. Finally, the teacher may reset the rounding to 10 decimal places and reset the slider of θ as interval from 89° to 91° and increment as 0.00001° . Then, the teacher may drag the slider of θ slowly and ask the students to observe how the values of the trigonometric ratios $y(B')$ and $x(C)$ change. It is expected that the students would

realize the sign of $\tan\theta$ changes at 90° , and there is a rapid increase on the magnitude of $\tan\theta$ when θ is moved towards 90° and then a rapid decrease on the magnitude of $\tan\theta$ when θ is moved away 90° (Figure 3). Thus, the behaviour of tangent function near $\theta = 90^\circ$ can be discussed. It provides a dynamic illustration to the concept of limit of a function. The above discussion extends the learning objective from the definitions of trigonometric functions to the behaviours of the functions, and hence broadens the learning area from trigonometry to calculus. This extension seems to be impossible without the mistake opportunity provided by the discrepancy of the software. It relies on teacher's sensitivity in realizing such an opportunity and making use of it to initiate mathematics discussion.

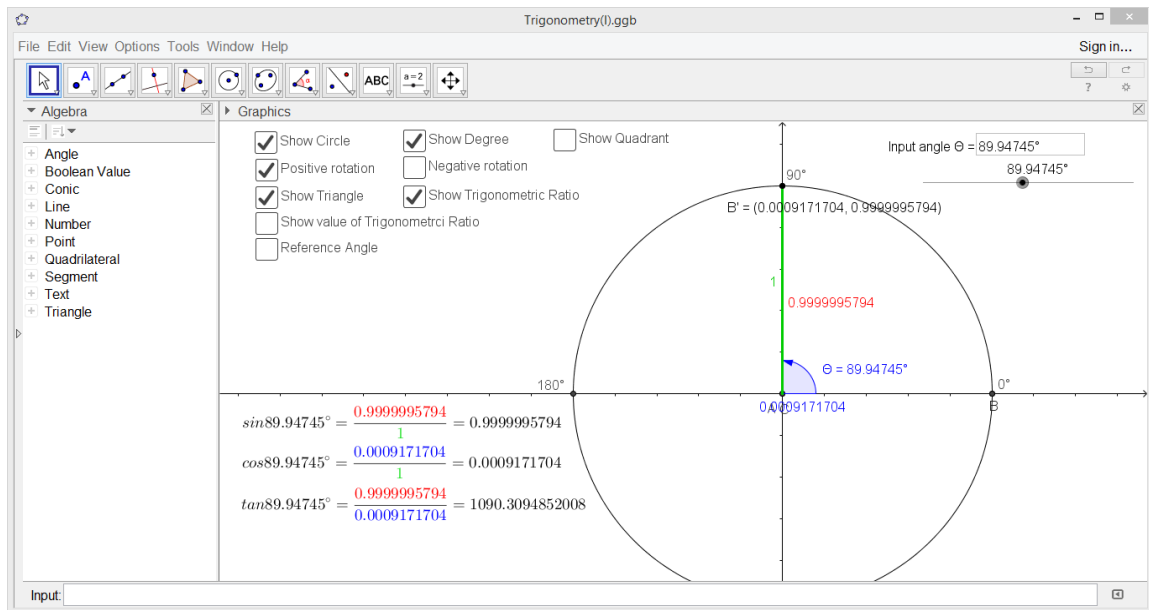


Figure 3: Zooming the behaviour of tangent function near 90 degrees

3.2 Case 2: Tangent to a circle

There are many properties related to the tangents to a circle. Some of them are not easy to be understood. One of the properties is that a tangent-chord angle of a circle equals to an angle in the alternate segment (usually abbreviated as: theorem of angle in alternate segment). Peter constructed a GeoGebra figure to illustrate this property (Figure 4) [S3]. While dragging point P along the circle, $\angle a$ (tangent-chord angle) keeps equal to $\angle b$ (angle in the alternate segment). However, problem arises when point P crosses point K . In that case, the orientation of the triangle is changed and hence $\angle a$ is no longer equal to $\angle b$. This phenomenon relates to the continuous fidelity of the software. It may be regarded as an embedded discrepancy of the software. Yet, it highlights a common misconception of the students and thus has potential to correct students' misunderstanding. For a tangent to the circle and a chord, there are two tangent-chord angles in which they are supplementary to each other (that is, their sum equals to 180°). Students may have difficulty in identifying which tangent-chord angle corresponding to which angle in the alternate segment. The discrepancy caused by the change of orientation of the triangle can enhance students' abilities in identifying the correct angle pair and deepen their understanding on the orientation of a geometric figure which is an important but often overlooked issue.

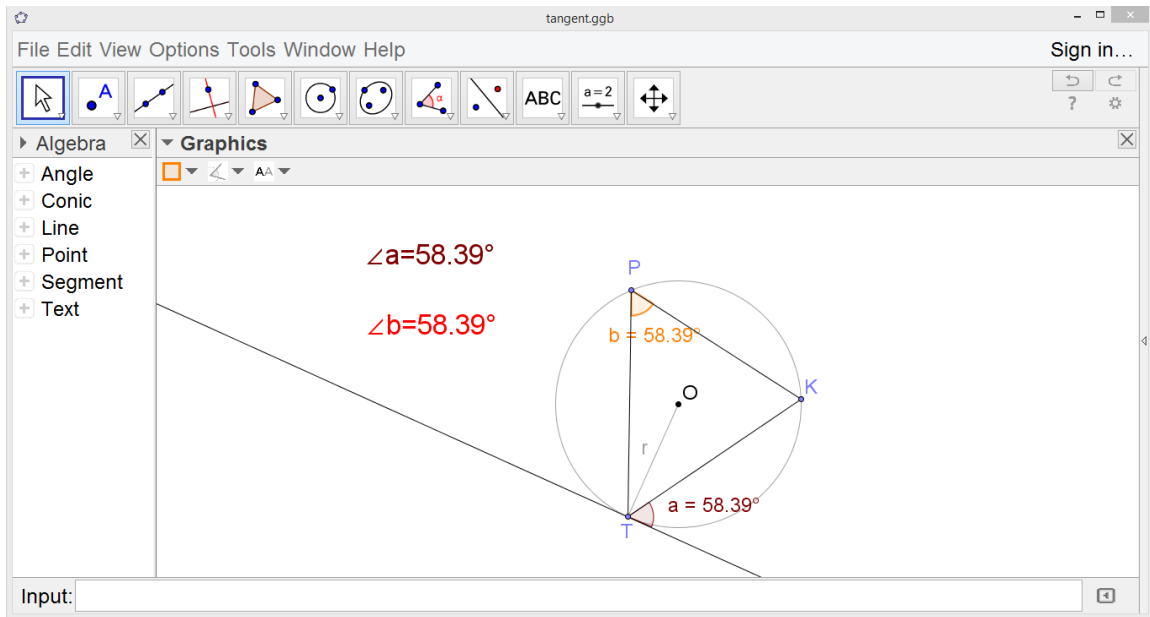


Figure 4: *Theorem of angle in alternate segment*

In the following, we will propose how mathematics discussions can be arranged so that this embedded discrepancy can be transformed to an enacted discrepancy and brought out the learning opportunity. First, the teacher can drag P along the circle and ask the students to put attention on the values of $\angle a$ and $\angle b$. While point P moves along the circle, $\angle TPK (= \angle b)$ changes whereas its value keeps unchanged. (This is the property of angles in the same segment of a circle.) When P reaches a position where PT passes through the center O (that is, PT is a diameter), $\angle a$ becomes 90° and hence $\angle b + \angle \gamma = 90^\circ$. It is also noted that $\angle a + \angle \gamma = 90^\circ$ because of the radius-tangent perpendicularity (Figure 5). It gives a (visual) proof for the theorem of angle in alternate segment.

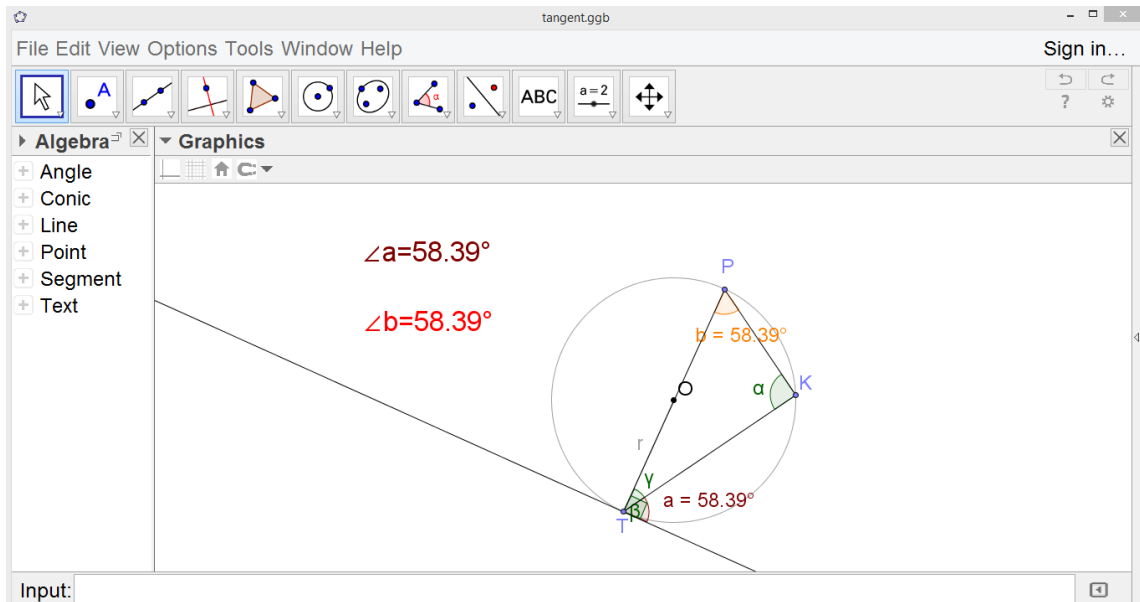


Figure 5: *Angle in alternate segment for a right-angled triangle*

After explaining the proof of the theorem, the teacher can continue to drag P along the circle to make the discrepancy visible (and thus making the discrepancy enacted). After point P crosses point K , the teacher can point out that $\angle a \neq \angle b$ and asks the students what has happened

(Figure 6). The teacher can ask the students why the above proof is not applicable. By observing the diagram, it is expected that the students can realize that $\angle a$ becomes a reflexive angle (that is, it is greater than 180°) and hence the orientation of $\triangle TPK$ is changed from clockwise to anti-clockwise. Thus, $\angle TPK (= \angle b)$ does not correspond to the tangent-chord angle $\angle a$ any more. The teacher can further ask the students to find the relation between the value of $\angle a$ and the value of $\angle b$. It should be easily find that $\angle a + \angle b = 180^\circ$. Indeed, $\angle a$ corresponds to the supplementary angle of $\angle b$ in the cyclic quadrilateral QKPT (Figure 6). It is hoped that this explorative activity can help students to recognize the correct angle pair for the theorem of angle in alternate segment and have a deeper understanding to the orientation of a triangle.

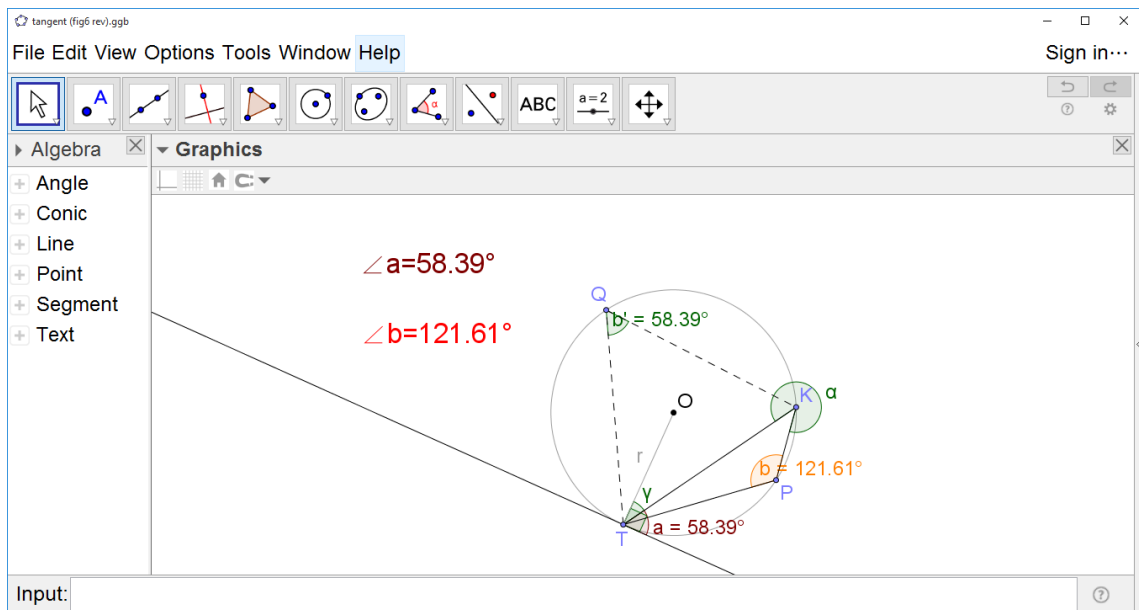


Figure 6: Change of orientation when point P crosses point K

4. Discussion

In the following sections, we will propose a general strategy for using the notion of discrepancy potential to teach mathematics and then discuss how to prepare teachers to utilize this notion in their own teaching. Finally, research agenda related to this notion will be proposed.

4.1 General strategy for using the notion of discrepancy potential to teach mathematics

The definition of discrepancy potential (quoted in Section 2) points out two types of sources of discrepancy potential in which pedagogical spaces can be generated. The first type is the deviation of feedback from the tool and the intended mathematical concept to be taught. The second type is the uncertainty created due to the tool or the task. The two cases described in Section 3 illustrates two situations of the first type of discrepancy potential. The first case deals with extreme situations of a mathematical concept. In this case, they are the boundaries of the quadrants of a coordinate plane. The second case deals with common misconceptions and overlooking aspect of a mathematical concept. In this case, it is the orientation of a geometric figure. (Examples for the second type of discrepancy potential can be found in [3] in which DGS soft construction tasks are discussed.) After the teacher introduces the basic definition of a mathematical concept and the students have worked out enough consolidation exercises, the teacher can discuss with the students the extreme situations of the mathematical concept through ‘pedagogical tasks with discrepancy potential’. A natural way for transiting from the standard situations to the extreme situations is to

ask “what-if” questions. The teacher may ask the students to predict what would happen if the parameter reaches to the extreme situations. The extreme situations may also be included in the consolidation exercises. Instead of aiming at getting correct answers, it aims at initiating further discussion on these extreme situations. The teacher may instruct the students to ‘test’ these extreme situations using the software package with embedded discrepancy. In the usual method, software package gives correct answers and thus it serves for answer-checking only. Higher order argument and reasoning are unlikely to occur. However, this innovative teaching method provides ‘surprise’ for generating pedagogical spaces which can motivate teacher-led argument and reasoning. The teacher can use the normal situations which are ‘near’ to the extreme situations as an analogue and ask the students to decide whether they believe the answers provided by the computer. It is expected that a debate would be initiated. As illustrated in Case 1 of Section 3, the teacher can make use of the built-in functions of the software to enable a close look at the extreme situations and compare them with the normal situations. After deliberated class discussion, the teacher can sum-up with a rigor argument to the extreme situations under investigated.

Dealing with misconception through ‘pedagogical tasks with discrepancy potential’ can be implemented in review part of a mathematical topic. Similar as above, the teacher can instruct the students to make guess before using the software package with embedded discrepancy to demonstrate the common mistakes, as illustrated in Case 2 of Section 3. Then, the teacher can invite the students to point out the abnormal phenomenon (or mistakes) arising from the discrepancy. It would open a pedagogy space for discussing and highlighting the common misconceptions.

Although computer software (when it is used appropriately) can facilitate mathematics learning, some scholars raised the concern about the ‘death’ of reasoning and proving because of introducing computer software in mathematics classrooms (see for instance, [10, 11, 12]). However, the use of discrepancy potential proposed above may be able to resolve this dilemma because computer can ‘make mistakes’ and hence it is no longer used for ‘answer-checking’ only. It becomes a means for initiating mathematics discussion. Thus, computer is no longer regarded as an authority in which students can rely on without the need of mathematical thinking. It is believed that this teaching approach matches with the mission of eJMT of “demonstrating how mathematics and its applications can be made interesting, accessible and challenging – all without compromising mathematical rigor”.

4.2 Preparing teachers to utilize the notion of discrepancy potential

It has been pointed out in Section 2 that discrepancy does not necessary can enhance students’ learning. If it is utilized inappropriately, discrepancy can hinder students’ learning. It is emphasized in [4] that “teacher’s sensitivity and insight in identifying the discrepancy opportunity and discrepancy pitfall (both embedded and enacted) and the ability to adjust the instrumental length are key factors in successfully exploiting the semiotic potential of the tool-based task” (p.281). And thus, it is essential to prepare teachers to have such pedagogical abilities. The first step for achieving this uneasy goal is to help the teachers become familiar with the affordance and constraint of different types of software and help them realize how the software represents the mathematical concepts intend to be taught. Teachers need to be sensitive enough in identifying the distinction in representing the same mathematical concept by different software, no matter how subtle the distinction is. In fact, even whether continuity fidelity preserves can make a great different. As a way for developing such abilities, teachers may try to compare different software of similar type. Through exploring how different software behave in different geometric situations especially those extreme situations, it is hoped that the teachers can realize the distinctions in the discrepancies of different software and develop the ability in determining whether the discrepancies are opportunities or pitfalls.

4.3 Research agenda related to the notion of discrepancy potential

Discrepancy potential is a young and still developing theoretical notion. Several research agenda related to this notion can be proposed. First, as pointed out by researchers of ICT in mathematics education (see for instance [13]), teachers play a crucial role to the success of technology-based mathematics tasks and didactical activities. One of the key factors is teachers' sensitivities in identifying discrepancy potential and their abilities in distinguishing whether it is a discrepancy opportunity or a discrepancy pitfall. It is acknowledged that the teaching scenarios described above are only hypothetical. Whether it can really happen depends on teachers' competences in identifying and handling the discrepancy potentials. Therefore, teaching experiments using the examples described in this paper (or some other similar examples) in real classrooms can be conducted. It is not just a matter of whether these examples can enhance the learning effectiveness or not. It is more important to investigate how these discrepancies facilitate or hinder mathematical learning. In other words, it is vital to identify the guiding principles for distinguishing a discrepancy opportunity from a discrepancy pitfall. Second, a survey on teachers' (or pre-service teachers') readiness in seeing the pedagogical potentials of discrepancies (or more generally technological constraints) can be conducted. Discrepancy examples such as those described in this paper can be showed to the teachers for comment. If they think that there is pedagogical potential, we may ask them to design a lesson based on these examples. We may also ask the teachers to suggest other discrepancy examples which are beneficial to students' learning. This type of survey which is a kind of need analysis can inform the planning for preparing teachers (and pre-service teachers) to utilize discrepancy potential. Third, putting the research agenda in the area of teacher education and teacher knowledge, it is worthwhile to investigate how teachers' competences in enlarging the discrepancy opportunity and diminishing the discrepancy pitfall are related to teachers' technology pedagogical content knowledge [14], their beliefs about mathematics and mathematics teaching as well as their conceptions towards the role of technology in mathematics education. We believe that these research questions can help us to positioning the notion of discrepancy potential into the theoretical frameworks of IT in mathematics education and other related research areas. It is hoped that this paper can enlighten the readers to perceive the technological affordances and constraints from an alternative perspective and thus open-up a new 'window' for designing and researching IT-based mathematics learning tasks.

5. Conclusion

Computer program such as DGS has its own design constraints. The constraints may lead to deviations from the intended mathematical concepts. These deviations have potentials to provide pedagogical opportunities and hence are called discrepancy potentials. In this paper, the discrepancy potentials of two DGS files are discussed. The discrepancies are due to rounding off errors (Case 1) and the change of orientation (Case 2) respectively. The discussion above illustrated how these discrepancies can be used for deepening students' mathematical understanding and clearing up possible misconceptions.

Discrepancy potential is a young and still developing theoretical notion. The author and his collaborators are working on other empirical studies to test its powerfulness in researching tool-based mathematics pedagogy tasks. Yet, it is evidenced that a new window to understand the efficiency of technology tools in teaching mathematics has been opened up.

Acknowledgement

The author would like to thank the two pre-service mathematics teachers for giving consents to use their files.

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Software package

[GeoGebra] GeoGebra 5, developed by Markus Hohenwarter and his team 2001/2016, <http://www.geogebra.org>

Supplementary files

[S1] S1_Trigo.ggb, a *GeoGebra* file created by a pre-service teacher Jane (pseudonym), <https://www.geogebra.org/m/xdtQnTWa>

[S2] S2_Trigo(revised).ggb, a revised version of Jane's *GeoGebra* file (modified by the author), <https://www.geogebra.org/m/ugK76S8M>

[S3] S3_Tangent.ggb, a *GeoGebra* file created by a pre-service teacher Peter (pseudonym), <https://www.geogebra.org/m/YjTu752y>